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Pervasive Function Spaces and the Best Harmonic Approximation

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1. Introduction

Suppose that $U \subset \mathbb{R}^m$ is a bounded open set with boundary ∂U. Denote by $H(\partial U)$ the set of all functions on ∂U for which there is a solution of the classical Dirichlet problem. Thus $f \in H(\partial U)$ provided f has a continuous extension to the closure $\bar{U}$ of U which is harmonic on U. It is known that $H(\partial U)$ is a uniformly closed subspace of the Banach space $C(\partial U)$ of all continuous functions on ∂U. In general, however, $H(\partial U) \neq C(\partial U)$. Thus given $f \in C(\partial U)$, one may try to find amongst the functions of $H(\partial U)$ the best uniform approximation to f. The aim of this note is to investigate the possibility of such an approximation. It turns out that, in a typical case, the space $H(\partial U)$ is a pervasive function space. This motivates our investigation of the question of the best approximation by elements of pervasive spaces.

2. Pervasive Spaces and the Best Approximation

Let X be a compact Hausdorff topological space and $C(X)$ be the sup-norm space of all continuous real valued functions on X. By a function space (on X) we mean a closed subspace of $C(X)$. For $F \subset X$ closed and a function space L, the symbol $L_{|F}$ denotes the set of all restrictions of the functions of L to the set F.

A function space L is called pervasive provided the following condition is satisfied: Whenever F is a nonempty proper closed subset of X, then $L_{|F}$ is dense in $C(F)$.

The dual space $C^*(X)$ of $C(X)$ will be, as usual, identified with the space of real Borel regular signed measures on X. The (closed) support of a measure $\mu \in C^*(X)$ will be denoted by spt μ.

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For a function space $L \subset C(X)$, $L^{\perp}$ denotes the annihilator of L, i.e., the subspace of $C^*(X)$ consisting of all μ such that $\int f \, d\mu = 0$ whenever $f \in L$.

It is shown in [5] that a space L is pervasive if and only if the support of any nontrivial measure in $L^{\perp}$ is all X.

Suppose that L is a function space. For $f \in C(X)$ denote

$$P_L(f) = \{ g \in L;\ \|g - f\| = \inf\{ \|f - h\|;\ h \in L \} \},$$

$$B_L = \{ f \in C(X) \setminus L;\ P_L(f) \neq \emptyset \}.$$

The space L is said to be

(a) proximinal, if $B_L = C(X) \setminus L$,

(b) Čebyšev, if $P_L(f)$ contains exactly one point for every $f \in C(X)$;

(c) very non-proximinal, if $L \neq C(X)$ and $B_L = \emptyset$;

(d) almost very non-proximinal, if $L \neq C(X)$ and B_L is of the first category in $C(X)$.

The best approximation in $C(X)$ is studied in detail in [14]; cf., in particular pp. 33, 117, 313.

Whenever $f \in C(X)$, the symbol $\mathrm{Lin}(L, f)$ stands for the linear span of $L \cup \{f\}$.

Let K_1, K_2 be closed subsets of X. Then K_1, K_2 are said to be L-separated, if there is a function $h \in L$ such that $h > 0$ on K_1 and $h < 0$ on K_2.

Denote

$$Q = \{ g \in C(X);\ g(X) \subset \{-1, 1\} \}$$

and for $g \in Q$ put

$$A^+(g) = g^{-1}(\{1\}), \qquad A^-(g) = g^{-1}(\{-1\}).$$

A function $g \in C(X)$ is called a Q_L-function, if $g \in Q$ and $A^+(g)$ and $A^-(g)$ are not L-separated.

THEOREM 1. *Let L be a pervasive function space and $f \in C(X) \setminus L$. Then $f \in B_L$ if and only if there is a Q_L-function g such that $f \in \mathrm{Lin}(L, g)$.*

Proof. Let $f \in B_L$ and $h \in P_L(f)$. By the Hahn-Banach theorem, there exists $\mu \in L^{\perp}$ such that $\|\mu\| = 1$ and

$$\int (f - h) \, d\mu = \|f - h\|.$$

Since $\|f - h\| = \int (f - h) \, d\mu = |\int (f - h) \, d\mu| \leqslant \int |f - h| \, d|\mu| \leqslant \|f - h\|$, we have $\|f-h\| = |f-h|$ $|\mu|$-almost everywhere. The function $|f-h|$ is continuous and $\operatorname{spt}|\mu| = \operatorname{spt}\mu = X$ because L is pervasive. Consequently,

$$|f-h| = \|f-h\|$$

everywhere on X. Hence there is $g \in Q$ such that

$$f-h = \|f-h\| \cdot g.$$

Thus $f \in \operatorname{Lin}(L, g)$ and we are going to show that $A^+(g)$ and $A^-(g)$ are not L-separated. Indeed, the assumption that $A^+(g)$ and $A^-(g)$ are L-separated implies the existence of $h_1 \in L$ such that $h_1 > 0$ on $A^+(g)$ and $h_1 < 0$ on $A^-(g)$. For a suitable $a \in \mathbb{R}$ we have $\|g - a \cdot h_1\| < 1$, thus

$$\|(f-h)/\|f-h\| - a \cdot h_1\| < 1.$$

Putting $h_0 = h + a \cdot \|f-h\| \cdot h_1$, we get $h_0 \in L$ and $\|f-h_0\| < \|f-h\|$. This is impossible, since $h \in P_L(f)$. We conclude that g is a Q_L-function.

Conversely suppose that g is a Q_L-function such that $f \in \operatorname{Lin}(L, g)$. Thus there is $c \in \mathbb{R}$ and $h \in L$ such that $f = h + c \cdot g$. Clearly, $c \neq 0$ since $f \notin L$. We are going to show that $h \in P_L(f)$. Assume that there is $h_1 \in L$ such that $\|f-h_1\| < \|f-h\|$. Then $\|c \cdot g + h - h_1\| < |c|$ and for $h_0 = c^{-1}(h_1 - h) \in L$ we have $\|g - h_0\| < 1$. Then $h_0 > 0$ on $A^+(g)$ and $h_0 < 0$ on $A^-(g)$, which means that $A^+(g)$ and $A^-(g)$ are L-separated. This contradicts the hypothesis that g is a Q_L-function. Thus $h \in P_L(f)$ and $f \in B_L$.

Proposition 1. *Suppose that L is a pervasive function space and* $\operatorname{codim} L = 1$. *Then L is either a Čebyšev space or a very non-proximinal space. If $\mu \in L^\perp \setminus \{0\}$, then L is a Čebyšev space if and only if $\operatorname{spt}\mu^+ \cap \operatorname{spt}\mu^- = \emptyset$. If X is connected and $L \setminus \{0\}$ contains a positive function, then L is very non-proximinal.*

Proof. Fix $\mu \in L^\perp$ such that $\|\mu\| = 1$ and write $K^+ = \operatorname{spt}\mu^+$, $K^- = \operatorname{spt}\mu^-$. Since L is pervasive, $X = K^+ \cup K^-$.

If both K^+ and K^- are non-empty and $g \in C(X)$, $g \neq 0$, then $|\int g\, d\mu| < \|g\|$. Thus if $K^+ \cap K^- \neq \emptyset$, then $|\int (f-h)\, d\mu| < \|f-h\|$ whenever $f \in C(X) \setminus L$ and $h \in L$. Since $\dim L^\perp = 1$, we conclude easily that L is very non-proximinal.

Assume now that $K^+ \cap K^- = \emptyset$ and denote $m^+ = \mu(K^+)$, $m^- = -\mu(K^-)$. Then $m^+ \geqslant 0$, $m^- \geqslant 0$ and $m^+ + m^- = 1$. We may suppose that $m^- > 0$ since otherwise we could consider $-\mu$ instead of μ.

Define $g = 0$ on K^+, $g = 1$ on K^-. Then $g \in C(X)$, $\int g\, d\mu = -m^- < 0$. Thus $g \notin L$ and $C(X) = \operatorname{Lin}(L, g)$. Put $h_0 = m^-$ on K^+ and $h_0 = m^+$ on K^-. Then

$$\int h_0\, d\mu = m^+ \cdot m^- - m^- \cdot m^+ = 0.$$

It follows easily that $h_0 \in L$ and $\|g - h_0\| = m^-$. Let $h \in L$ and $\|g - h\| \leqslant m^-$. Then $h \leqslant m^-$ on K^+ and $h \geqslant 1 - m^- = m^+$ on K^-. If $h(x) < m^-$ for some $x \in K^+$ or $h(x) > m^+$ for some $x \in K^-$, then

$$0 = \int h \, d\mu = \int h \, d\mu^+ - \int h \, d\mu^- < m^- m^+ - m^+ m^- = 0.$$

Consequently, $h = m^-$ on K^+ and $h = m^+$ on K^-, i.e., $h = h_0$.

One easily verifies that

$$\inf\{ \|g - h\| ; h \in L \} = |\mu(g)| = m^- .$$

We conclude that g has the unique best approximation h_0. If $f \in C(X) \setminus L$, then $f = h_1 + cg$ for suitable $h_1 \in L$ and $c \in \mathbb{R} \setminus \{0\}$. Clearly, f has the unique best approximation $h_1 + ch_0$. This means that L is a Čebyšev subspace.

Let X be connected. Then $\operatorname{spt} \mu^+ \cap \operatorname{spt} \mu^- = \emptyset$ implies that $\operatorname{spt} \mu^+ = X$ or $\operatorname{spt} \mu^- = X$. This is impossible provided $L \setminus \{0\}$ contains a positive function. Consequently, L is then very non-proximinal.

THEOREM 2. *Suppose that X is metrizable and L is a pervasive function space such that* $\operatorname{codim} L > 1$. *Then L is almost very non-proximinal. If, moreover, X is connected and L contains the constant functions, then L is very non-proximinal.*

Proof. Notice that $\|g_1 - g_2\| = 2$, whenever $g_1, g_2 \in Q$, $g_1 \neq g_2$. Since X is metrizable, the space $C(X)$ is separable, whence Q is countable. If $g \in C(X)$, then $\operatorname{Lin}(L, g)$ is a closed subspace of $C(X)$, since L is a closed subspace; cf. [13, p. 87]. Since $\operatorname{codim} L > 1$, $\operatorname{Lin}(L, g) \neq C(X)$ and thus $\operatorname{Lin}(L, g)$ is a nowhere dense subset of $C(X)$. By Theorem 1,

$$B_L \subset \bigcup \{ \operatorname{Lin}(L, g); g \in Q \},$$

hence B_L is of the first category and L is almost very non-proximinal.

If X is connected, then Q contains exactly two constant functions. Thus if L contains constant functions, then, by Theorem 1, $B_L = \emptyset$ and L is very non-proximinal.

3. WHEN IS $H(\partial U)$ A PERVASIVE SPACE?

In what follows the dimension m of $\mathbb{R}^m$ is supposed to be $\geqslant 2$. For $x \in \mathbb{R}^m$ and $r > 0$ denote $B_r(x) = \{ y \in \mathbb{R}^m; |y - x| \leqslant r \}$; λ stands for the m-dimensional Lebesgue measure and ε_x is the Dirac measure concentrated at x. For a compact set $M \subset \mathbb{R}^m$, $\lambda_{|M}$ denotes the restriction of λ to M.

As in the introduction, let U be a bounded open set in $\mathbb{R}^m$, $H(\partial U)$ be the space of all functions on ∂U having a continuous extension to $\bar{U}$ which is harmonic on U. For $x \in \partial U$, ε_x^{CU} stands for the balayaged measure of ε_x on $CU = \mathbb{R}^m \backslash U$; see [2, p. 75]. We have $\varepsilon_x^{CU} = \varepsilon_x$ if and only if x is a regular boundary point of U. Moreover, $f \in H(\partial U)$ if and only if $f \in C(\partial U)$ and $f(x) = \int f \, d\varepsilon_x^{CU}$ whenever $x \in \partial U$; see [11], cf., also [2, p. 99].

The space $H(\partial U)$ is not pervasive in general. Let e.g. $U = U_1 \cup U_2$, where U_1, U_2 are nonempty open, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and ∂U_1 contains an irregular point x. Then (considered as a measure on ∂U) $\varepsilon_x - \varepsilon_x^{CU_1} \in H(\partial U)^{\perp} \backslash \{0\}$ and $\operatorname{spt}(\varepsilon_x - \varepsilon_x^{CU_1}) \subset \partial U_1 \neq \partial U$.

But even for a domain U, $H(\partial U)$ need not be pervasive. Take an open bounded domain $V \subset \mathbb{R}^m$ with exactly one irregular point $x \in \partial V$, fix $y \in V$ and put $U = V \backslash \{y\}$. Then $\mu = \varepsilon_x - \varepsilon_x^{CU} \in H(\partial U)^{\perp} \backslash \{0\}$ and $y \notin \operatorname{spt} \mu$.

One may ask whether $H(\partial U)$ is pervasive, provided U is a domain such that $\partial U \cap W$ is of positive capacity, provided W is an open set such that $\partial U \cap W \neq \emptyset$. The answer is negative, as shown by the following example due to Hansen.

Let V be a bounded open set in $\mathbb{R}^m$ having the following properties: V contains $B_1(0)$, is symmetric with respect to the hyperplane $M = \{(x_1, ..., x_m) \in \mathbb{R}^m; x_1 = 0\}$ and there are exactly two distinct irregular points $x, y \in \partial V$ symmetrical with respect to M. Put $U = V \backslash (M \cap B_1(0))$. Then x, y are (symmetrical) irregular points of U. Put $\mu = \varepsilon_x - \varepsilon_x^{CU} - (\varepsilon_y - \varepsilon_y^{CU})$. Then $\mu \in H(\partial U)^{\perp} \backslash \{0\}$ and a symmetry argument shows that the measures ε_x^{CU} and ε_y^{CU} coincide on $M \cap B_1(0)$. Consequently, $\mu \in H(\partial U)^{\perp} \backslash \{0\}$ and $\operatorname{spt} \mu \neq \partial U$.

In this example, $\partial U \neq \partial \bar{U}$ and we shall show below that, for a domain U with $\partial U = \partial \bar{U}$, the space $H(\partial U)$ is always pervasive.

(Note in this connection that Hansen constructed (private communication) an elliptic harmonic space (see [4]) and a relatively compact domain U with $\partial U = \partial \bar{U}$ such that an analogously defined space $H(\partial U)$ is not pervasive. On the other hand this cannot occur in potential theory associated to a wide class of elliptic partial differential operators; see the remark below.)

THEOREM 3. *Let $U \in \mathbb{R}^m$ be a bounded domain satisfying $\partial U = \partial \bar{U}$. Then the space $H(\partial U)$ is pervasive.*

Proof. Let $V = \mathbb{R}^m$, if $m > 2$ and V be a circle containing $\bar{U}$, if $m = 2$. Let G: $V \times V \to [0, \infty]$ be the Green function on V (cf. [2 or 9]). Let $\mu \in H(\partial U)^{\perp}$ and $\operatorname{spt} \mu \neq \partial U$. We are going to show that $\mu = 0$.

Fix a point $z \in \partial U$ and $r > 0$ such that $B_r(z) \cap \operatorname{spt} \mu = \emptyset$. Since $\partial U = \partial \bar{U}$, $B_r(z) \cap (V \backslash \bar{U}) \neq \emptyset$. Fix $x \in V \backslash \bar{U}$. The function $y \mapsto G(x, y)$ is harmonic on a neighbourhood of $\bar{U}$, thus $\int G(x, y) \, d\mu(y) = 0$. In other words, the functions $G\mu^+ : x \mapsto \int G(x, y)\, d\mu^+(y)$, $G\mu^- : x \mapsto \int G(x, y)\, d\mu^-(y)$ coincide on $V \setminus \bar{U}$. Since $\operatorname{spt} \mu \cap B_r(z) = \emptyset$, the functions $G\mu^+$ and $G\mu^-$ are harmonic on $B_r(z)$ and coincide, as shown above, on $B_r(z) \setminus \bar{U} \neq \emptyset$. But harmonic functions are real analytic (see, e.g. [2, p. 163]), thus we have $G\mu^+ = G\mu^-$ on $B_r(z)$, thus on $B_r(z) \cap U \neq \emptyset$. Since U is connected, the same analyticity argument shows that $G\mu^+ = G\mu^-$ on U. We conclude that the two potentials $G\mu^+$, $G\mu^-$ coincide on $V \setminus \partial U$. Now fix $x \in \partial U$, $\rho > 0$ and put $\nu = \lambda_{|\partial U \cap B_\rho(x)}$. Then $G\nu$ is continuous on V (cf. [9, p. 119]) and harmonic on U. Consequently,

$$\int G\nu\, d\mu^+ = \int G\nu\, d\mu^- .$$

By symmetry of G we have

$$\int G\mu^+\, d\nu = \int G\mu^-\, d\nu.$$

Since $G\mu^+ = G\mu^-$ on $V \setminus \partial U$ we get

$$\int_{B_\rho(x)} G\mu^+\, d\lambda = \int_{B_\rho(x)} G\mu^-\, d\lambda.$$

But $G\mu^+(x) = \lim_{\rho \to 0} (\lambda(B_\rho(x)))^{-1} \int_{B_\rho(x)} G\mu^+\, d\lambda$ and analogously for $G\mu^-(x)$; cf. [9, p. 70]. We conclude that $G\mu^+ = G\mu^-$ everywhere on V, which yields $\mu^+ = \mu^-$ by [9, p. 112]. Thus $\mu = 0$.

Remark. A similar reasoning can be used to establish an analogous assertion in the situation that solutions of an elliptic partial differential equation are considered instead of harmonic functions. The main difference in the proof is that, in view of non-symmetry of the Green function, one has to consider potentials corresponding to the adjoint equation. For relevant results from potential theory suitable for this more general situation see [1, 10, 12, 14].

4. The Best Harmonic Approximation

We shall suppose that U and $H(\partial U)$ have the same meaning as in Section 3.

Proposition 2. *Let U be a domain and ∂U contain exactly one irregular point. Then $H(\partial U)$ is pervasive and the following assertions holds:*
$H(\partial U)$ is a Čebyšev space if and only if ∂U has exactly one isolated point.
$H(\partial U)$ is very non-proximinal if and only if ∂U has no isolated points.

Proof. Note that every isolated point of ∂U is irregular. Let x be the only irregular point of U. Recall that $f \in H(\partial U)$ if and only if $f \in C(\partial U)$ and $f(x) = \int f \, d\varepsilon_x^{CU}$. It is known (see [6, p. 111]) that spt $\varepsilon_x^{CU} \supset \overline{\partial U \setminus \{x\}}$.

Fix $f_0 \in C(\partial U) \setminus H(\partial U)$ and for $f \in C(\partial U)$ put $a_f = (\int f \, d\varepsilon_x^{CU} - f(x))/(\int f_0 \, d\varepsilon_x^{CU} - f_0(x))$. Then

$$\int (f - a_f \cdot f_0) \, d\varepsilon_x^{CU} = f(x) - a_f \cdot f_0(x).$$

Thus $f - a_f \cdot f_0 \in H(\partial U)$ and codim $H(\partial U) = 1$.

If $\mu \in H(\partial U)^{\perp} \setminus \{0\}$, then there is $k \neq 0$ such that $\mu = k \cdot (\varepsilon_x^{CU} - \varepsilon_x)$. We conclude that spt $\mu = \partial U$ and $H(\partial U)$ is pervasive.

The assertions follow by Proposition 1.

THEOREM 4. *Let U be a domain, $\partial U = \partial \bar{U}$, and ∂U has at least two irregular points. Then $H(\partial U)$ is almost very non-proximinal. If ∂U is connected, then $H(\partial U)$ is very non-proximinal.*

Proof. Let x, y be different irregular points of U. Then $\varepsilon_x - \varepsilon_x^{CU}$, $\varepsilon_y - \varepsilon_y^{CU}$ are linearly independent elements of $H(\partial U)^{\perp}$. Thus codim $H(\partial U) > 1$. By Theorem 3, $H(\partial U)$ is a pervasive space. The rest follows from Theorem 2.

Remark. Other aspects of the best harmonic approximation have been investigated in [3, 7, 8].

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